① Notation: k = field. $\mathcal{O} = k[[t]]$ $F = \mathrm{Frac}(\mathcal{O}) = \mathcal{O}[\frac{1}{t}] = k((t))$.

Aff_k = category of affine k schemes with flat topology.

Defn. An ind-scheme $/k$ is a sheaf of sets of the form

$$\mathcal{F} = \operatorname{colim}_i X_i \quad \left(\begin{array}{l} X_i \text{ are schemes}/k, \text{ which is a} \\ \text{sheaf on } \mathrm{Aff}_k \text{ via Yoneda} \end{array} \right)$$

with transition = closed immersions.

Examples (1) a scheme over k ~~[illegible]~~

(2) a formal scheme over k.

[$\mathcal{F}_{red} = \operatorname{colim}_i (X_i)_{red}$, and $\mathcal{F}$ is a formal scheme $\Leftrightarrow$ $\mathcal{F}_{red}$ is a scheme]

Affine Grassmannians.

Defn. • R = k-algebra. An R-family of lattices in $k((t))^n$ is a f.g. proj. $R[[t]]^n$-submodule $\Lambda \leq R((t))^n$ s.t. $\Lambda \otimes [\frac{1}{t}] \cong R((t))^n$.

• Gr is the functor assigns $R \longmapsto \left\{ \begin{array}{c} R\text{-family of lattices} \\ \text{in } k((t))^n \end{array} \right\}$

Thm. The affine Grassmannian Gr is an $\underbrace{\text{ind-scheme}}_{\text{proj.}}$.

$$\left(Gr = \operatorname{colim}_i X_i, \ \text{each } X_i \subseteq \mathbb{P}_k^{N_i} \right).$$

Idea: Any Λ must be squeezed between $t^N \Lambda_0 \leq \Lambda \leq t^{-N} \Lambda$ where $\Lambda_0 = R[[t]]^n$ is the standard lattice.

So $Gr = \operatorname{colim}_N Gr^{(N)}$ those R-lattices squeezed

Lemma. $Gr^{(N)}$ is identified with

$$R \longmapsto \left\{ \text{quotients of } \frac{t^{-N} R[[t]]^n}{t^N R[[t]]^n} \text{ that are} \right\}$$

via $\Lambda \longmapsto t^{-N} R[[t]]^n / \Lambda$

① proj as R-mod and

② $\cdot t$ stable

Granting this lemma.

$$Gr^{(N)} \subseteq Gr\left(\frac{t^{-N}R[[t]]^n}{t^N R[[t]]^n}\right)$$

cut out by 2, which is a closed condition.

Now we need to prove this Lemma.

Step 1: Show $Q := t^{-N}R[[t]]^n/\Lambda$ is R-proj.

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] & R((t))^n/\Lambda \arrow[r] & R((t))^n/t^{-N}R[[t]]^n \arrow[r] & 0 \\
& & \bigoplus\limits_{k\geq 0} t^{-k-1}\Lambda/t^{-k}\Lambda \arrow[u, phantom, "\cong" rotate=90] & \text{proj.}/R \arrow[u] \arrow[l]
\end{tikzcd}$$

Step 2: need to show, ~~given Q a quotient as in the lemma~~
$\forall (R, Q)$, $\Lambda := \ker\left(t^{-N}R[[t]]^n \to \frac{t^{-N}R[[t]]^n}{t^N R[[t]]^n} \to Q\right)$
$\subseteq R((t))^n$

is finite proj. $/R[[t]]$.

$Gr^{(N)}$ is ~~locally~~ finite type $/k$, we reduce to R/k f.t.

Then $R[t] \longrightarrow R[[t]]$ is flat as $R[t]$ is Noetherian.

define $\Lambda_f := \ker\left(t^{-N}R[t]^n \longrightarrow \frac{t^{-N}R[t]^n}{t^N R[t]^n} \cong \frac{t^{-N}R[[t]]^n}{t^N R[[t]]^n} \longrightarrow Q\right)$,

$$\Lambda = \Lambda_f \otimes_{R[t]} R[[t]].$$

So suffice to show Λ_f is finite proj.

f.g. ✓ as submod. of $t^{-N}R[t]^n$.

already flat $/R$, so suffices to show flatness after modulo max'l ideals $\mathfrak{m} \subseteq R$.

$\Lambda_f/\mathfrak{m} = \ker(t^{-N}\kappa[t]^n \longrightarrow Q \otimes_R \kappa)$

③ flatness follows from being torsion free.

Transition $Gr^{(N)} \longrightarrow Gr^{(N+1)}$ is ~~easy~~ easily seen to be a closed immersion

Ind-projective.

canonical ample line on Gr:

$$(\Lambda_1, \Lambda_2) \in Gr \times Gr\,(R).$$

$$\det(\Lambda_1|\Lambda_2) := \det(\Lambda_1 / t^N R[[t]]^n) \otimes \det(\Lambda_2 / t^N R[[t]]^n)^{-1} \in Pic(R).$$

where $N >> 0$. (independent of N up to canonical isom.)

Moreover, for any $\Lambda_1, \Lambda_2, \Lambda_3$, $\exists$ canonical isom.

$$\gamma_{123} : \det(\Lambda_1|\Lambda_2) \otimes \det(\Lambda_2|\Lambda_3) \cong \det(\Lambda_1|\Lambda_3).$$

s.t. given $\Lambda_1, \dots, \Lambda_4$,

$$\gamma_{134}\gamma_{123} = \gamma_{124}\gamma_{234}.$$

Conclusion: • Get gpoid $\mathcal{L}_{det} \setminus$ zero-section $\rightrightarrows Gr$.

or $\mathbb{G}_m$-gerbe $\mathcal{D}_V = [Gr / \mathcal{L}^{\times}_{det}]$

• $\mathcal{L}_{det} / Gr \times Gr$. fix pt $\in Gr$ corresponds to the standard lattice.

$\rightsquigarrow$ ample $\mathcal{L}_{det}$ on $Gr \times pt \cong Gr$.

L^+GL_n & LGL_n

Defn. X presheaf over $\mathcal{O} = k[[t]]$. $L^nX(R) = X(R[t]/t^n) \subseteq (\mathrm{Res}_k^{R[t]/t^n} X)(R)$.

$$L^+X = \lim_n L^nX.$$

- Y presheaf over $F = k((t))$

$$LY(R) = Y(R((t))).$$

Prop. (1) X scheme over $\mathcal{O} = k[[t]]$.

Then $L^+X \dashrightarrow L^nX \longrightarrow L^1X \simeq X \times_{\mathcal{O}} k$ • representable.

is ① affine morphism,

② f. type $\forall\, n \in \mathbb{N}$.

③. open/closed $X \longrightarrow X'$ induce open/closed $L^+X \to L^+X'$.

(2) Y f.t. affine over $F = k((t))$.

Then LY is an ind-affine scheme$/k$.

$Y \longrightarrow Y'$ closed, $\rightsquigarrow$ $LY \longrightarrow LY'$ closed.

(3) $X/\mathcal{O}$ then $L^+X \hookrightarrow LX_F$ closed embedding.

Example: (1) $X = \mathbb{A}^1_{\mathcal{O}}$, then $L^+X(R) = R[[t]] = (\mathrm{Spec}\, k[\underset{\uparrow}{a_0}, \underset{\uparrow}{a_1}, \cdots])(R)$.

coefficients.

$$LX_F = \mathrm{colim}_i\ \mathrm{Spec}\, k[a_{-i}, a_{-i+1}, \cdots].$$

(2) $X = \mathbb{G}_{m/\mathcal{O}}$, then $L^+X(R) = \{(f = \sum a_i t^i,\ g = \sum b_j t^j) \in R[[t]]^2 \mid f \cdot g = 1\}$

$\Longrightarrow$ $L^+X = \mathrm{Spec}\, k[a_0^{\pm 1}, a_1, a_2, \cdots]$.

$$L_{-1}X_F(R) = \left\{ \begin{pmatrix} f = a_{-1}t^{-1} + a_0 + a_1 t + \cdots \\ g = b_{-1}t^{-1} + b_0 + b_1 t + \cdots \end{pmatrix} \;\middle|\; f \cdot g = 1 \right\}$$

$$\begin{cases} a_{-1} b_{-1} = 0 \\ a_{-1} b_0 + a_0 b_{-1} = 0 \\ a_{-1} b_1 + a_0 b_0 + a_1 b_{-1} = 1 \\ \quad \vdots \end{cases}$$

can check $a_0 b_{-1}^2 = a_{-1}^2 b_0 = 0$.

but $a_0 b_{-1} \neq 0$.

So non-reduced.

L^+X inside is cut out by setting $a_{-1} = b_{-1} = 0$.

⑤. Zhu claims that it's easy to see that LX is highly <u>non-reduced</u>. I don't (easily) see why...

~~In any~~ We're interested in $L^+GL_n \subseteq LGL_n$

$LGL_n(R) \circlearrowright Gr(R)$.

$g \in GL_n(R((t)))$, $\Lambda \subseteq R((t))^n$, $g \cdot \Lambda \subseteq g \cdot R((t))^n = R((t))^n$.

Hence we have $LGL_n \times Gr \longrightarrow Gr$.

if we take $\Lambda_0 = R[[t]]^n$ the standard lattice,

its stabilizer in LGL_n is exactly L^+GL_n!

Prop. $[LG/L^+G] \cdot \Lambda_0 \xrightarrow{\cong} Gr$.

↖ fpqc sheafify of the quotient presheaf.

pf. Λ f. proj. $/R[[t]]$, Λ/t is f. proj $/R$.

$\rightsquigarrow \exists\ R \to R'$ faithfully flat (or faithful étale).

s.t. $\Lambda/t \otimes_R R' \simeq R'^{\oplus n}$

Lift generators $\rightsquigarrow \Lambda \otimes_{R[[t]]} R'[[t]] \simeq R'[[t]]^{\oplus n}$.

$R'((t))^n \supseteq$

compare this with the standard lattice $R'[[t]]^n \subseteq R'((t))^n$.

gives $g \in GL_n(R((t)))$.

Cor. Gr is formally smooth, i.e. if $\tilde{R} \twoheadrightarrow R$ is a square zero thickening then $Gr(\tilde{R}) \twoheadrightarrow Gr(R)$.

Recall: $\mathcal{L}_{det} / Gr \times Gr$. $\quad \mathcal{L}_{det}(\Lambda_1, \Lambda_2) = \det(\Lambda_1 | \Lambda_2)$.

$g \in L(GL_n)$ $\quad \rho_g : \det(g.\Lambda_1 | g.\Lambda_2) \cong \det(\Lambda_1 | \Lambda_2)$.

satisfy cocycle condition.

$\rightsquigarrow LGL_n \curvearrowright \mathcal{L}_{det} / Gr \times Gr$.

Convolution Grassmannian: $Gr \tilde{\times} Gr (R) = (\Lambda_1 \subseteq R((t))^n$ lattice, ~~and $\Lambda_2 \subseteq \Lambda_1[\frac{1}{t}]$ lattice~~ Λ_2 f. proj $/R[[t]]$, with $\Lambda_2[\frac{1}{t}] \xrightarrow[\beta]{\sim} \Lambda_1[\frac{1}{t}]$)

Defn.

$LGL_n \times Gr$

$(g, \Lambda) \longmapsto g.\Lambda_0, \; \Lambda \longrightarrow R((t))^n \xrightarrow{g^{-1}} R((t))^n$

induces $LGL_n \times Gr / L^+GL_n \xrightarrow{\cong} Gr \tilde{\times} Gr$

diagonal left.

$(h\Lambda_0, \Lambda_2, \beta)$, with $\Lambda_1 = h\Lambda_0$

$(h, \Lambda_2 \hookrightarrow \Lambda_2[\frac{1}{t}] \xrightarrow{\beta} \Lambda_1[\frac{1}{t}] \cong R((t))^n)$

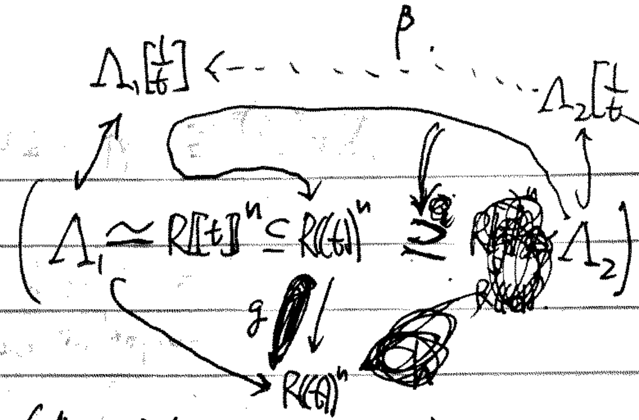
conceptually better: $\Lambda \in Gr(R)$ s.t. Λ is trivial.

$GL_n(R[[t]])$ -worth of ambiguity.

$$\begin{array}{ccc} \Lambda & \cong & R[[t]]^n \subseteq R((t))^n \\ \cap & & \\ R((t))^n & \leftarrow & \exists! \, g \in GL_n(R((t))) \text{ making comm.} \end{array}$$

part of data of Λ being an R-pt of Gr.

⑦ X $\begin{cases} LGL_n \times Gr \\ (g, \; \Lambda_2 \subseteq R((t))^n) \longmapsto \end{cases}$



$g\cdot, \; g\beta \longleftarrow \left(\begin{array}{l} \Lambda_1 \text{ R-lattice,} \\ \Lambda_2[\frac{1}{t}] \xrightarrow[\simeq]{\beta} \Lambda_1[\frac{1}{t}] \end{array}\right).$

Prop. gives $LGL_n \overset{L^+GL_n}{\times} Gr \cong Gr \tilde{\times} Gr$.

$\Lambda_1 \subseteq R((t))^n$ R-lattice.

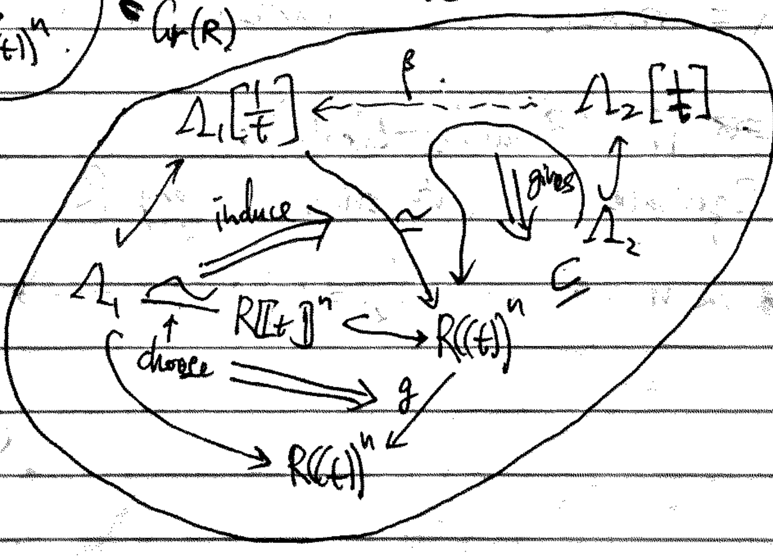
g. $\qquad \Lambda_2$ f. proj $/R[[t]]$ w/

$\beta: \Lambda_2[\frac{1}{t}] \xrightarrow{\simeq} \Lambda_1[\frac{1}{t}]$

$\Lambda_2 \hookrightarrow \Lambda_2[\frac{1}{t}] \xrightarrow{\beta} \Lambda_1[\frac{1}{t}] \simeq R((t))^n \in Gr(R)$

choose $\Lambda_1 \simeq R[[t]]^n$

hence $\Lambda_1[\frac{1}{t}] \simeq R((t))^n$



Extra symmetry: $Aut(D)$. assigns $R \longmapsto$

$Aut(D)(R) = Aut_{R, cont}(R[[t]])$.

$\varphi(t) = a_0 + a_1 t + a_2 t^2 + \cdots \in R[[t]]$

induces a continuous auto iff a_0 is nilpotent and a_1 is invertible.

$$\mathrm{Aut}(D) \cong \mathrm{Spf}\, k[\![a_0]\!] \times \mathrm{Spec}\, k[a_1^{\pm 1}, a_2, a_3, \cdots] \twoheadrightarrow \mathrm{Spf}\, k[\![a_0]\!]$$

closed normal $\cup|$ subgp.

$$\mathrm{Aut}^+(D) \cong \mathrm{Spec}\, k[a_1^{\pm 1}, a_2, \cdots] \longrightarrow \mathrm{Spec}\, k[a_1^{\pm 1}] =: \mathbb{G}_m^{\mathrm{rot}}$$

$\cup|$ pro-unipotent radical. rotation torus.

$$\mathrm{Aut}^{++}(D) \cong \mathrm{Spec}\, k[a_2, a_3, \cdots]$$

$$\mathrm{Aut}(D) \times \mathrm{Gr} \longrightarrow \mathrm{Gr}.$$

$$(\varphi, \ \Lambda \subseteq R(\!(t)\!)^n) \longmapsto (\Lambda \otimes_{R[\![t]\!], \varphi} R[\![t]\!] \subseteq R(\!(t)\!)^n \otimes_{R(\!(t)\!), \varphi} R(\!(t)\!) \cong R(\!(t)\!)^n.$$

$$\left(\text{or} \quad (\varphi, \ g \cdot L^+GL_n) \longmapsto \varphi(g) \cdot L^+GL_n\right).$$

Beauville–Laszlo's moduli interpretation.

X = reduced connected curve$/k$ with $x \in |X|$ smooth closed pt.

Thm (Beauville–Laszlo):

Equivalence of categories for any R

$$\{\text{rk } n \text{ v.b. on } X_R := X \otimes_k R\} \longleftrightarrow \left\{\begin{array}{l} ① \text{ rk } n \text{ v.b. on } D_{x,R} \\ ② \text{ rk } n \text{ v.b. on } X_R^* := (X \setminus \{x\}) \otimes_k R \\ ③ \ \varphi: \text{identification of these 2} \\ \quad \text{v.b.s restricted to } D_{x,R}^* \end{array}\right\}$$

$$(X_R)^\wedge_{x \times R} = D_{x,R}, \qquad D^*_{x,R} =: D_{x,R} \times_{X_R} X_R^*$$

Cor: LGL_n represents the functor

$R \longmapsto \{(\mathcal{E}, \alpha, \beta)$, where $\mathcal{E}$ is a rk n v.b. on X_R, $\alpha: \mathcal{E} \times_{X_R} D_{x,R} \simeq \bigoplus_n \mathcal{O}_{D_{x,R}}$, $\beta: \mathcal{E} \times_{X_R} X_R^* \simeq \bigoplus_n \mathcal{O}_{X_R^*}\}$.

⑨

pf sketch: to get $\mathcal{E}$ from α & β, by B–L Thm, we need to specify $\varphi: \mathcal{O}_{D^*_{x,R}}^{\oplus n} \xrightarrow{\cong} \mathcal{O}_{D^*_{x,R}}^{\oplus n}$

which is exactly $GL_n(R((t))^n)$.

as $D^*_{x,R} = \mathrm{Spec}\,(R((t)))$.

$$g \longleftrightarrow \left(\beta|_{D^*_{x,R}}\right) \circ \left(\alpha|_{D^*_{x,R}}\right)^{-1}.$$

Now ~~quotienting~~ if $g \in L^+GL_n$, then we may absorb g in the ~~identifying~~ trivialization α.

So ~~L^+GL_n is~~ quotienting out L^+GL_n amounts to forgetting the trivialization α. Hence we get

Thm: Gr represents the functor

$$R \longmapsto \{(\mathcal{E}, \beta) \text{ where } \mathcal{E} \text{ is a rk } n \text{ v.b. on } X_R,$$
$$\beta: \mathcal{E} \times_{X_R} X_R^* \cong \bigoplus_n \mathcal{O}_{X_R^*}\}$$